

# Stochastic dynamics of an electron in a Penning trap: phase flips correlated with amplitude collapses and revivals

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## Abstract

We study the effect of noise on the axial mode of an electron in a Penning trap under parametric-resonance conditions. Our approach, based on the application of averaging techniques to the description of the dynamics, provides an understanding of the random phase flips detected in recent experiments. The observed correlation between the phase jumps and the amplitude collapses is explained. Moreover, we discuss the actual relevance of noise color to the identified phase-switching mechanism. Our approach is then generalized to analyze the persistence of the stochastic phase flips in the dynamics of a cloud of  $N$  electrons. In particular, we characterize the detected scaling of the phase-jump rate with the number of electrons.

## 1 Introduction

The research on electron traps has opened the way to significant advances in fields ranging from Atomic Physics to Metrology [1]. For instance, the application of the trapping techniques has been crucial for achievements like the generation of antimatter atoms [2] or the realization of precision tests on fundamental constants [3]. Moreover, trapped electrons provide a controllable testing ground for a variety of physical behaviors, predicted or experimentally identified in other areas. Actually, the possibility of controlling the trapping setup, in particular, of varying its components, can allow the systematic characterization of different effects via their realization under well-defined conditions and in regimes unexplored in other contexts. In this way, problems like the emergence of nontrivial effects of noise [4], the preparation of Fock states [5], the

appearance of squeezing in quantum-dissipation processes in nonlinear oscillators [6, 7, 8], or the implementation of proposals for quantum-information algorithms [9, 10, 11, 12] have been analyzed with different variations of the basic trapping setup. Here, we focus on a novel effect detected in recent experiments on electrons in a Penning trap [4]. Namely, under parametric-resonance conditions, the axial mode of a one-electron system was observed to present random amplitude collapses correlated with phase flips. That behavior was traced to noise rooted in different elements of the practical arrangement. Indeed, by adding fluctuations in a controlled way, the dependence of the phase-jump rate on the noise strength was characterized. Remarkably, for increasing noise intensities, the correlation between the amplitude collapses and the phase jumps was found to disappear. The study of the persistence of those effects in the dynamics of a cloud of  $N$  electrons revealed a nontrivial behavior, in particular, the attenuation and eventual disappearance of the stochastic phase switching as the number of electrons was increased. Despite the advances in the characterization of the observed dynamics [4, 13], a satisfactory explanation of the underlying physical mechanisms is still needed, as stressed in Ref. [4]. For example, the actual relevance of colored noise to the emergence of some of the detected effects is an open question. Here, we present a description of the dynamics based on averaging techniques applicable to stochastic systems. From our approach, the origin of the experimental findings is uncovered and the elements of the system which are essential for the appearance of the observed behavior are identified. Furthermore, our semi-analytical characterization of the dynamics provides us with some clues to controlling the system response to the fluctuations. The direct implications of the study to the advances in the techniques of confinement and stabilization are evident. Additionally, because of the fundamental character of the physics involved, the analysis can be relevant to different contexts, where conditions similar to those realized in the trapping scenario can be implemented [14, 15].

The outline of the paper is as follows. In Sec. II, we present our model for the stochastic dynamics of the axial mode of an electron in a Penning trap. Through the application of the averaging methods of Bogoliubov, Krylov, and Stratonovich [16, 17], we derive an effective description in terms of a system of stochastic differential equations for the amplitude and the phase. In Sec. III, the validity of our approach is confirmed through the simulation of the main experimental findings. Moreover, an analysis of the physical mechanisms responsible for the observed behavior is presented. Sec. IV contains the study of the persistence of the stochastic phase switching in the dynamics of a cloud of  $N$  electrons. Finally, some general conclusions are summarized in Sec. V.

## 2 The model system

An electron in a Penning trap is usually described in terms of three coupled modes (magnetron, axial, and cyclotron) with widely different time scales [1]. Here, we concentrate on the axial coordinate  $z$ . In general, because of the inter-

mode coupling, the axial dynamics can be quite complex: a variety of behaviors can emerge depending on the considered regime of experimental parameters. However, under standard conditions, different approximations can be made and the description simplifies considerably. Namely, the (slow) magnetron motion can be adiabatically treated and its influence on the dynamics can be reduced by sideband cooling [4]. Additionally, by damping the (fast) cyclotron mode to its fundamental state, its effect on the axial coordinate is minimized [4]. In typical realizations,  $z$  is coupled to a measuring external circuit, which introduces resistive damping and noise in the mode; moreover, the trap potential that is usually applied is approximately harmonic. Hence, in the basic scheme,  $z$  corresponds to a dissipative harmonic oscillator which can be described classically. Axial outputs qualitatively different from that basic realization can be generated by introducing different driving fields and controlled fluctuations in the practical setup [1, 18, 19, 20]. Apart from directly affecting the dynamics, those external elements can indirectly enhance the effect of the nonlinear corrections to the trap potential. Here, we aim at explaining the experiments of Ref. [4]. In them, a nontrivial axial dynamics was uncovered in a variation of the basic setup which incorporated a driving field at parametric resonance and noise of controllable intensity. Those experimental conditions are simulated in our approach. Specifically, we consider that  $z$  is parametrically driven at a frequency  $\omega_d$  which is nearly twice the characteristic resonant frequency  $\omega_z$ , i.e.,  $\omega_d = 2(\omega_z + \epsilon)$  with the restriction  $\epsilon \ll \omega_z$  for the detuning  $\epsilon$ . Moreover, our model incorporates a stochastic force  $\eta(t)$  with the characteristics of the noise present in the practical setup. Residual nonlinear terms of the confining potential, which are known to account for the stabilization of the noiseless version of the system in the parametric-resonance regime [18, 19], are also included in the model. Accordingly, we consider that the axial coordinate, (normalized to a typical trap length, and, therefore, dimensionless), is described by the equation

$$\ddot{z} + \gamma_z \dot{z} + \omega_z^2 [1 + h \cos \omega_d t] z + \lambda_4 \omega_z^2 z^3 + \lambda_6 \omega_z^2 z^5 = \eta(t), \quad (1)$$

where  $\gamma_z$  is the friction coefficient,  $h$  characterizes the amplitude of the driving force, and,  $\lambda_4$  and  $\lambda_6$  are coefficients which determine the magnitude of the nonlinear terms of the confining potential. By now, we deal with a one-electron system; later on, we will tackle the dynamics of a cloud of  $N$  electrons.

Crucial to the applicability of our model to the considered experiments is the appropriate modeling of the stochastic force  $\eta(t)$ . Two types of fluctuations are relevant to the experimental scheme. First, the system presents “internal” noise rooted in different elements of the practical setup. It has been argued that those fluctuations are well modeled by broadband noise and centered narrow-band fluctuations. Second, white noise, more intense than the internal fluctuations, was injected in the experimental realization of Ref. [4]. Indeed, it was the addition of this “external” noise that allowed studying the dependence of the switching mechanism on the noise strength. Here, in order to account for both, the broadband internal fluctuations and the added white noise, we consider that  $\eta(t)$  has general Gaussian wideband characteristics [21]. Specifically, we

assume that the correlation function  $k_\eta(t' - t) \equiv \langle \eta(t)\eta(t') \rangle - \langle \eta(t) \rangle^2$  has a generic functional form and that the correlation time is much shorter than any other relevant time scale in the system evolution. The intensity coefficient  $D = \frac{1}{2} \int_{-\infty}^{\infty} k_\eta(\tau) d\tau$  will be used to characterize the noise strength [17]. (The white-noise limit, defined by  $k_\eta(t' - t) = 2D\delta(t - t')$ , is included in our analysis.) Additionally, a zero mean value,  $\langle \eta(t) \rangle = 0$ , is assumed. (Notice that a nonzero  $\langle \eta(t) \rangle$  can be simply incorporated into the model as an effective deterministic contribution.) A more elaborate noisy input should be added to tackle the effect of residual colored noise. However, that generalization of our approach is not necessary for the objectives of the present paper: it will be shown that the detected behavior can be simply traced to broadband-noise characteristics.

Our approach to deal with Eq. (1) is based on the averaging methods developed by Krylov and Bogoliubov for the analysis of deterministic nonlinear oscillations as they were generalized by Stratonovich to the study of stochastic processes [16, 17]. Those averaging techniques can be applied to generic wide-band fluctuations with sufficiently short correlation time. In this approach, the amplitude  $A$  and the phase  $\Psi$  of the oscillations are defined through the equations  $z = A \cos[(\omega_z + \epsilon)t + \Psi]$  and  $\dot{z} = -(\omega_z + \epsilon)A \sin[(\omega_z + \epsilon)t + \Psi]$ . With these changes, Eq. (1) is reduced to a system of two first-order equations in *standard form* [17], i.e., with the structure of a harmonic oscillator perturbed by deterministic and stochastic terms. For  $\omega_z \gg \gamma_z, \epsilon$ , the average of the deterministic perturbative elements over the period  $\tau_{ef} = 2\pi/(\omega_z + \epsilon) = 4\pi/\omega_d$  is readily carried out. Moreover, for a noise correlation time much smaller than the relaxation times of the amplitude and the phase, the coarse graining of the stochastic terms over  $\tau_{ef}$  can be applied following the procedure presented in Ref. [17]. Accordingly, we obtain that, to first-order, the averaged equations are [17, 20, 22]

$$\dot{A} = -\frac{\gamma_z}{2} \left[ 1 - \frac{h}{h_T} \sin 2\Psi \right] A + \frac{D_{eff}}{A} + \xi_1(t), \quad (2)$$

$$\dot{\Psi} = -\epsilon + \frac{3}{8} \lambda_4 \omega_z A^2 + \frac{5}{16} \lambda_6 \omega_z A^4 + \frac{1}{4} \omega_z h \cos 2\Psi + \frac{\xi_2(t)}{A}, \quad (3)$$

where we have introduced  $h_T \equiv 2\gamma_z/\omega_z$ . (The meaning of  $h_T$  as a threshold amplitude of the driving field will be evident shortly.) Additionally,  $\xi_1(t)$  and  $\xi_2(t)$  are effective Gaussian white-noise terms defined by  $\langle \xi_i(t) \rangle = 0$ , and  $\langle \xi_i(t) \xi_j(t') \rangle = 2D_{eff} \delta_{i,j} \delta(t - t')$ ,  $i, j = 1, 2$ , with  $D_{eff} = \kappa_\eta(\omega_z + \epsilon)/[4(\omega_z + \epsilon)^2]$ . (Note that  $D_{eff}$ , which determines the strength of the (uncorrelated) effective noise terms, is obtained from the power spectral density  $\kappa_\eta(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega\tau} k_\eta(\tau) d\tau$  of the original noise  $\eta(t)$  at the frequency  $\omega_z + \epsilon$ . Here, we must remark that, from the broadband characteristics assumed for  $\eta(t)$ , a smooth form of  $\kappa_\eta(\omega)$  can be inferred. Indeed, a completely flat spectrum occurs in the white-noise limit.) Whereas the noise term in Eq. (2),  $\xi_1(t)$ , is additive, the fluctuations enter Eq. (3) through the term  $\xi_2(t)/A$ , and, therefore, have multiplicative character. Moreover, it is important to take into account the presence of the noise-induced “deterministic” term  $D_{eff}/A$  in Eq. (2): its appearance

will be shown to account for the partial character of the amplitude collapses detected in the experiments. It is worth emphasizing that our use of averaged equations is specially appropriate for the considered experimental setup, where, because of the specific characteristics of the detection scheme, the registered data do actually correspond to averaged magnitudes.

In order to trace the response of the system to noise, we must clearly define the deterministic scenario into which the fluctuations enter. The noiseless dynamics of the system, described by Eq. (1) without the random term  $\eta(t)$ , and, consequently, by Eqs. (2) and (3) with  $D_{eff} = 0$ , has been intensively studied [18, 19]. From the averaged equations, it is straightforwardly shown that parametric amplification, i.e., exponential growth of the amplitude, takes place for a driving amplitude  $h$  larger than the threshold value  $h_T$  and for a detuning  $\epsilon$  within the excitation range, namely, for  $\epsilon_- < \epsilon < \epsilon_+$  ( $\epsilon_{\pm} = \pm \frac{\omega_z}{4} \sqrt{h^2 - h_T^2}$ .) The experimental conditions on which we focus correspond to this parametric-amplification regime. In the absence of nonlinear terms in the trap potential, the amplitude would grow monotonously. However, the nonlinear corrections, characterized by the coefficients  $\lambda_4$  and  $\lambda_6$ , which must be included in the description to simulate the actual potential applied in the practical setup, do arrest the amplitude growth, allowing the stabilization of the motion. The system presents two stationary states with the same amplitude and with phase values differing in  $\pi$  radians. Specifically, the stationary amplitude  $A_{SS}$  is obtained from the equation  $\epsilon_+ - \epsilon + \frac{3}{8}\lambda_4\omega_z A_{SS}^2 + \frac{5}{16}\lambda_6\omega_z A_{SS}^4 = 0$ , and the two  $\pi$ -differing values of the equilibrium phase  $\Psi_{SS}$  are given by  $\Psi_{SS} = \frac{1}{2} \arcsin(\frac{h_T}{h})$ . [Important for the discussion of some of the noisy features is to notice that, for the values of the nonlinear parameters applied in the experiments, ( $\lambda_4 = 0$  and  $\lambda_6 < 0$ ),  $A_{SS}$  increases with  $\epsilon_+ - \epsilon$ .] The stationary states correspond to attractors in the phase space. Depending on the initial conditions, the system eventually reaches one or other attractor. The objective of the next section is the explanation of the effects of noise on this deterministic scenario.

### 3 Stochastic dynamics of a one-electron system

In the analysis of the noisy dynamics, we proceed by showing first that Eqs. (2) and (3) provide a satisfactory description of the behavior observed in the experiments. Then, once its validity has been confirmed, our approach will be applied to uncover the physics underlying the detected features.

#### 3.1 The response to noise

In Ref. [4], the presence of noise was shown to significantly alter the deterministic picture. The system was not longer stabilized in one of the attractors. Instead, it was observed to display amplitude collapses and revivals correlated with abrupt changes in the phase. Those experimental findings are reproduced by our approach. Figs. 1a and 1b respectively depict results for  $A$  and  $\Psi$  as obtained from Eqs. (2) and (3). There, the correlation between the phase flips

and the collapses and revivals of the amplitude is evident. Actually, in agreement with the experimental results, it is found that the amplitude collapses are almost always followed by phase flips. In other words, the system rarely stays in the same basin of attraction once a collapse in  $A$  has occurred. One should notice that, in the inter-jump intervals,  $\Psi$  is strongly localized around its equilibrium values; in contrast, a significant dispersion in  $A$  is observed. Our study reproduces the detected partial character of the collapses. Already noticeable in Fig. 1a, this feature is particularly evident in Fig. 2, where we depict a typical noisy trajectory, which includes different flips between the two basins. There, it is apparent that the unstable point defined by  $A = 0$  is never reached. As stressed in Ref. [4], these features cannot be understood with a simple activation-process model.

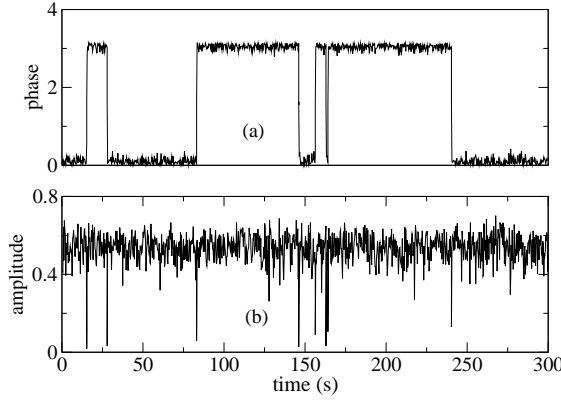


Figure 1: Time evolution of the phase (a) and of the amplitude (b) as obtained from Eqs. (2) and (3). [ $\omega_z/2\pi = 61.6MHz$ ,  $\lambda_4 = 0$ ,  $\lambda_6 = -0.27$ ,  $\gamma_z = (10ms)^{-1}$ ,  $\epsilon_+/2\pi = 100Hz$ ,  $\epsilon/2\pi = 50Hz$ ,  $D = 10^{-2}$  (arbitrary units).] (The same set of parameters is used throughout the paper.)

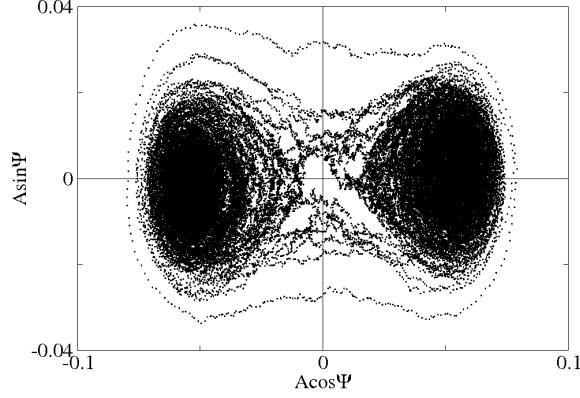


Figure 2: Phase space diagram for a particular noisy trajectory.

In the analysis of the experimental results, a histogram for the residence time, (i.e., the time interval between phase flips), served to obtain the average jump rate  $\Gamma$ . Actually,  $\Gamma$  was found to be well approximated by an exponential function of the noise strength, namely,  $\Gamma \sim \exp(-E/D)$ , where  $E$  denotes the effective activation energy. That characterization is reproduced by our approach. In Fig. 3, we represent a histogram for the residence time as obtained from Eqs. (2) and (3). Moreover, in Fig. 4, we plot the jump rate as a function of the noise strength. There, the validity of the exponential fit is patent. (In our calculations, we have directly worked with  $D_{eff}$  as noise strength: the used arbitrary units include the ratio between the actual strength of the original noise  $D$  and  $D_{eff}$ .) Apart from the dependence on the fluctuations, the jump rate incorporates, through the effective activation energy, the influence on the process of elements like the frequency and strength of the driving field, the characteristics of the trapping potential, or the damping coefficient. The description of the role played by those deterministic components of the system in the noisy dynamics is crucial for understanding the process of phase switching. In the experiments, the dependence of  $E$  on those system parameters was traced via the systematic variation of the practical conditions. In particular, an approximately exponential dependence of the jump rate  $\Gamma$  on the detuning, expressed as  $\epsilon_+ - \epsilon$ , was reported. That behavior is also simulated with our approach, as shown in Fig. 5.

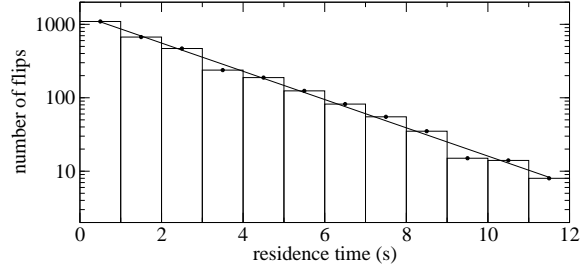


Figure 3: Histogram for the time interval between phase flips.

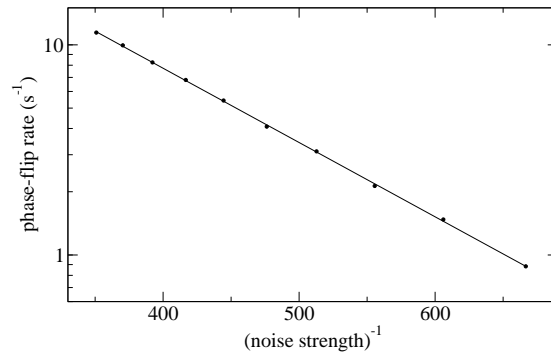


Figure 4: Phase-jump rate  $\Gamma(s^{-1})$  versus the inverse noise-strength  $D^{-1}$  (arbitrary units).



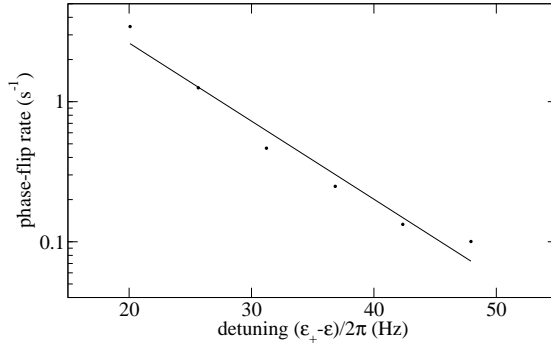


Figure 5: Phase-jump rate  $\Gamma(s^{-1})$  as a function of the detuning expressed as  $(\epsilon_+ - \epsilon)/2\pi$  (Hz). ( $\epsilon_+/2\pi = 100Hz$ ).

In Ref. [4], no results are presented for the dependence of the activation energy on the friction coefficient. However, as this aspect of the system behavior will be an important element of our discussion of the phase switching mechanism, it is pertinent to describe it here using our approach. Indeed, since early theoretical studies were set up from a Hamiltonian approximation to the dynamics, it is worthwhile to inquire into the actual significance of the dissipative character of the system. Let us first recall some aspects of the purely deterministic (dissipative) dynamics which are relevant to the resulting stochastic scenario. Namely, from the found threshold amplitude, given by  $h_T = 2\gamma_z/\omega_z$ , it is evident that the generation of oscillations is inhibited as the friction coefficient  $\gamma_z$  increases. Additionally, we must take into account that, as found in Ref. [11], the relaxation of the system from any initial conditions to the equilibrium states becomes faster for larger  $\gamma_z$ . Therefore, we can conjecture that, as  $\gamma_z$  grows, simply because of the deterministic inhibition of the oscillations and of the enhanced stability of the system, the role of noise in activating the phase jumps must be hindered. This conjecture is confirmed by our results for the dependence of  $E$  on  $\gamma_z$ . We have found that there is an approximately exponential decrease of the jump rate  $\Gamma$  with  $\gamma_z$ , as shown in Fig. 6. Consequently, from the expression  $\Gamma \sim \exp(-E/D)$ , a nearly linear increase of  $E$  with  $\gamma_z$  is derived. Specifically, we can write  $E \approx C_1 + C_2\gamma_z$ , where the coefficients  $C_1$  and  $C_2$  incorporate the dependence of  $E$  on other parameters of the system. Some implications of these results will be considered in the forthcoming discussion. By now, we anticipate that the analysis of the persistence in the  $N$ -electron system of the found dependence of  $E$  on  $\gamma_z$  will be central to our understanding of the detected scaling of the phase jump rate with  $N$ .

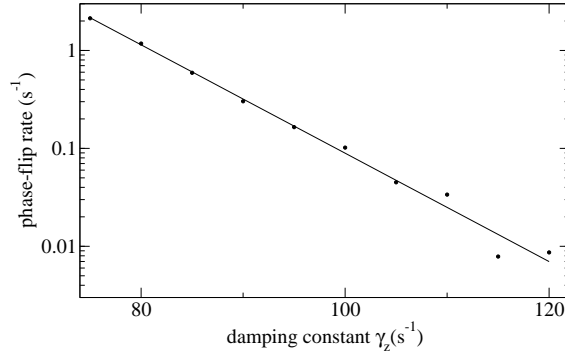


Figure 6: Phase-jump rate  $\Gamma(s^{-1})$  versus the friction coefficient  $\gamma_z$  ( $s^{-1}$ ).

### 3.2 The mechanism of stochastic phase switching

In our discussion of the physics that underlies the observed features, we proceed gradually: we start with a simplified picture of the dynamics, which will be improved by successively incorporating the different elements of the complete system. Our scheme is summarized in the following steps.

(i) A zero-order approximation to the random response detected in the experiments is provided by the artificial decoupling of Eqs. (2) and (3). Indeed, some clues to the origin of prominent features of the complete system are given by the analysis of the “independent” behaviors of  $A$  and  $\Psi$  that respectively follow from fixing the phase in Eq. (2) and the amplitude in Eq. (3).

For a constant value of  $\Psi$ , Eq. (2) describes fluctuations of  $A$  around its equilibrium position. Interestingly, because of the term  $D_{eff}/A$ , the equilibrium amplitude is larger than its counterpart in the absence of noise. Noise-induced excursions to the region of small  $A$  can be predicted. Due to the effective “deterministic” term  $D_{eff}/A$ , the value  $A = 0$  is not reached, i.e., a complete collapse never takes place. As in the description of the deterministic system at parametric resonance, to account in our approach for the limited growth of  $A$  observed in practice, the coupling to the phase equation, which contains the nonlinear terms of the confining potential, is necessary.

Conversely, for a fixed  $A$ , Eq. (3) describes a process of phase diffusion in a tilted periodic potential [17]. The bias, given by  $-\epsilon + \frac{3}{8}\lambda_4\omega_z A^2 + \frac{5}{16}\lambda_6\omega_z A^4$ , is determined by the detuning and by the artificially fixed amplitude. The potential presents two minima separated by  $\pi$  radians. When the bias is smaller than the height of the periodic potential,  $\Psi$  evolves only because of the fluctuations. Actually, in the regime considered in the experiments, it is noise that

leads to phase jumps between the minima. Since the magnitude of the (multiplicative) random term  $\xi_2(t)/A$  increases as  $A$  diminishes, the jumps become more frequent for smaller  $A$ .

(ii) A comparative analysis of the structure of the two averaged equations uncovers the qualitatively different effects of noise on the two variables. Since the amplitude has no strong confining potential, it is continuously forced out of equilibrium by the stochastic force. Indeed, a significant dispersion in  $A$  is apparent in Fig. 1a. In contrast, as shown in Fig. 1b, the periodic potential leads to a remarkable concentration of  $\Psi$  around its equilibrium values. This phase locking is interrupted by noise-induced flips. Attention must also be paid to some characteristics of the coupling between Eqs. (2) and (3). As previously discussed, the presence of  $A$  in Eq. (3) is crucial for the evolution of  $\Psi$ . In particular, the magnitude of  $A$  determines the frequency of the phase flips via the random term  $\xi_2(t)/A$ . In contrast, a much weaker effect of the phase flips on the evolution of the amplitude is apparent: given that the phase enters Eq. (2) through  $\sin 2\Psi$ , the  $\pi$  jumps in the phase hardly alter the dynamics of the amplitude.

(iii) By combining the ideas contained in the above points, the observed correlation between amplitude collapses and phase jumps can be explained. Noise can induce an appreciable reduction in the amplitude, which leads to a significant increase of the random term  $\xi_2(t)/A$  in the equation for the phase evolution, and, in turn, to a stochastic phase jump. Because of the “deterministic” term  $D_{eff}/A$ , the unstable point with  $A = 0$  and undefined  $\Psi$  is avoided in the switching. In fact, as  $A$  never reaches a zero value, the phase is always well-defined. Additionally, the fast regrowth of the amplitude after each (partial) collapse is rooted in the term  $D_{eff}/A$  and in the friction-induced stability of the underlying deterministic attractors. In the experiments, the correlation between the amplitude collapses and the phase jumps was observed to decay for increasing noise intensities. This feature can be understood taking into account that, as the noise strength increases, the random term  $\xi_2(t)/A$  can be strong enough to lead to phase flips even without an appreciable reduction in  $A$ . Furthermore, the inhibition of the jumps for increasing  $\gamma_z$  is linked to the higher stability of the (deterministic) stationary amplitude  $A_{SS}$  and to the consequent less probable exploration by the system of the small-amplitude region. Then, we can understand that, as shown in Fig. 6, the phase jumps dwindle as  $\gamma_z$  is enhanced, and, correspondingly, that the effective activation energy increases with  $\gamma_z$ . A similar argument qualitatively explains the dependence of the flip rate on the detuning, reflected in Fig. 5. As previously pointed out,  $A_{SS}$  grows with  $\epsilon_+ - \epsilon$ ; consequently, for increasing  $\epsilon_+ - \epsilon$ , the collapse region is less easily reached, and, in turn, the phase jumps become less probable.

From the above discussion, the essential components of the mechanism responsible for the stochastic phase-switching can be identified. The periodic potential, rooted in the driving field at parametric resonance, allows the strong localization of  $\Psi$ , and, therefore, the well-defined character of the jumps in phase. Additionally, the random term  $\xi_2(t)/A$  accounts for the correlation between amplitude collapses and phase flips. This stochastic link can be traced

to two fundamental characteristics of the system. First, it is rooted in the additive character of the input noise  $\eta(t)$ : with the change of representation, from  $z$  to  $A$  and  $\Psi$ , the fluctuations become multiplicative and the random connection does appear. Second, its specific compact form  $\xi_2(t)/A$ , uncovered by the application of the averaging methods, is a consequence of the broadband-noise characteristics, which guarantee the applicability of that methodology. Given the generality of its origin, this random link can be expected to be relevant to quite generic stochastic oscillators. In fact, it has been previously characterized in pioneering work on nonlinear self-excited oscillations in electronic devices [17]. Its intense differential effect on the current scenario results from its combination with the driving field: as the phase is strongly confined, its noise-induced evolution, (enhanced for small values of the amplitude), occurs basically through noticeable jumps between approximate equilibrium values. A comment on the role played by the nonlinearity of the system is also pertinent. It is important to emphasize that the nonlinear terms of the trap potential, which are necessary for the stabilization of the amplitude in the parametric-resonance regime, are not essential components of the phase-switching mechanism. It is the nonlinearity induced by the parametric driving field, i.e., the phase bi-stability, that really counts in the process. Also, it is interesting to examine what can be extracted from our approach about the actual relevance of colored noise. We recall that the possibility of tracing some of the observed features to noise-color characteristics was pointed out in early discussions of the experiments. In this sense, our study conclusively shows that the experimental features reported in Ref. [4] can be induced purely by broadband noise. Even more, we have found that simple white noise can account for those effects. Finally, we remark that, as stressed in Ref. [4], a simple activation-process model [21] fails to provide an appropriate picture of the observed dynamics. Although the introduction of an effective activation energy is useful in the characterization of the phase-jump rate, a simple activation-process description misses the sequence of combined effects in the evolution of the amplitude and phase that leads to the distinctive characteristics of the phase switching.

## 4 stochastic dynamics of a cloud of $N$ electrons

In the above, we have considered a mono-electronic system. Now, we turn to analyze the dynamics of a cloud of  $N$  electrons. We aim at explaining the nontrivial features of the evolution of the center-of-mass coordinate  $Z$  ( $Z = \frac{1}{N} \sum_{i=1}^N z_i$ ) uncovered by the experiments of Ref. [4], specifically, the observed slow-down and eventual disappearance of the random phase jumps for increasing number of electrons.

Some preliminary general considerations on the dynamics of the electronic cloud are in order. First, we recall that, in previous work on damping of a polyelectronic system in a Penning trap, the friction coefficient of  $Z$ ,  $\gamma_z^{(N)}$ , was shown to be well approximated as  $\gamma_z^{(N)} = N\gamma_z$  [18, 19]. Second, we must take into account that the random force on the center-of-mass coordi-

nate is given by  $\eta^{(N)}(t) = \frac{1}{N} \sum_{i=1}^N \eta_i(t)$ , where  $\eta_i(t)$  denotes the noise on each individual electron. The statistical characterization of  $\eta^{(N)}(t)$  is straightforward. As given by a linear superposition of Gaussian fluctuations,  $\eta^{(N)}(t)$  has also Gaussian-noise characteristics. From the zero-mean values of the mono-electronic stochastic forces, one trivially obtains  $\langle \eta^{(N)}(t) \rangle = 0$ . Additionally, assuming that the individual random forces are completely uncorrelated, i.e.,  $\langle \eta_i(t) \eta_j(t') \rangle = 2D\delta_{ij}\delta(t-t')$ ,  $i, j = 1 \dots N$ , we obtain  $\langle \eta^{(N)}(t) \eta^{(N)}(t') \rangle = 2\frac{D}{N}\delta(t-t') \equiv 2D^{(N)}\delta(t-t')$ . Hence, the individual fluctuations average to a weaker noise in the collective coordinate. (In order to present our arguments in simple terms, we have considered here the white-noise limit. The generalization to generic broadband noise is direct.) It follows that the main novelties in the description of  $Z$  with respect to the previously described mono-electronic scenario are the presence of a larger damping coefficient and of a reduced noise strength. Our analysis will focus on the implications of those differential characteristics for the scaling of the phase-flip rate with  $N$ . Since, as shown by the study of the one-electron oscillator, the mechanism of stochastic phase switching does not essentially depend on the anharmonicity of the potential, we can neglect the nonlinear terms in the analysis. Accordingly, we consider the evolution of  $Z$  as approximated by

$$\ddot{Z} + \gamma_z^{(N)} \dot{Z} + \omega_z^2 [1 + h \cos \omega_d t] Z = \eta^{(N)}(t). \quad (4)$$

Given that Eq. (4) has the same structure as Eq. (1), the methodology previously presented for the study of the mono-electronic system can also be used here. Indeed, this parallelism allows us to extrapolate some of the previous results. In particular, the flip rate, which, for a one-electron system was found to scale as  $\exp(-E/D)$ , can be expected now to have the form  $\exp(-E^{(N)}/D^{(N)})$ , where  $D^{(N)}$  is the (effective) strength of  $\eta^{(N)}(t)$  and  $E^{(N)}$  is the activation energy for  $Z$ . Crucial to the analysis is to take into account that, both,  $E^{(N)}$  and  $D^{(N)}$ , depend on  $N$ . Actually, as pointed out in the above application of our approach, the activation energy increases with the damping constant. Therefore, as the friction coefficient is  $\gamma_z^{(N)} = N\gamma_z$ , we conclude that  $E^{(N)}$  increases with  $N$ , and, consequently, that the phase flips are hindered as  $N$  grows. An additional contribution to the inhibition of the phase switching is rooted in the curbed fluctuations in the center-of-mass coordinate: the decrease of the effective noise strength  $D^{(N)} = D/N$  with  $N$  contributes also to the hindrance of the phase jumps for growing electron number. We can go further in the analytical characterization of the dependence of  $\Gamma^{(N)}$  on  $N$ : by combining the equations  $\Gamma^{(N)} \sim \exp(-E^{(N)}/D^{(N)})$ ,  $E^{(N)} = C_1 + C_2 N \gamma_z$ , and  $D^{(N)} = D/N$ , we obtain

$$\Gamma^{(N)} \sim \exp(-N(C_1 + C_2 N \gamma_z)/D). \quad (5)$$

This expression is the key element in our explanation of the observed scaling of the jump rate. The experimental procedure reported in Ref. [4] included different variations of the system parameters. Specially revealing of the mechanism responsible for the jump inhibition is the analysis of the experimental

run corresponding to a simultaneous variation of  $N$  and  $\gamma_z$  with constant  $N\gamma_z$ . Indeed, the observed decrease of the jump rate with  $N$  points to a mechanism not linked to the friction term  $\gamma_z^{(N)}$ , which, in fact, is kept constant in this run. From our approach, we can conjecture that, in this case, it is the decrease of the effective noise strength that leads to the detected slow-down of the phase flips. More specifically, the measured linear dependence of the exponent of  $\Gamma^{(N)}$  on  $N$  can be traced to the term  $NC_1/D$  in our analytical characterization of  $\Gamma^{(N)}$  given by Eq. (5). Additional insight is provided by the experimental results corresponding to a mere variation of  $N$ , with constant  $\gamma_z$ . In this case, both, a reduction in  $D^{(N)}$  and an increase of  $\gamma_z^{(N)}$  take place. Since, again, an approximately linear dependence of the exponent of  $\Gamma^{(N)}$  on  $N$  was found, we can conjecture that, in the regime studied in the experiments, the reduced noise strength is the dominant element in the mechanism responsible for the inhibition of the phase switching. Following the report of the experimental findings, our discussion in this section has focused on the persistence of the stochastic flips in the polyelectronic system. For a more complete description of the dynamics of the electronic cloud, the access to additional experimental data is necessary.

## 5 concluding remarks

Our description of the stochastic dynamics of the one-electron Penning-trap oscillator explains the experimental findings of Ref. [4]. The physical mechanism responsible for the observed random phase switching has been traced to the combination of a driving field at parametric resonance and broadband noise entering additively the axial-mode equation. The driving field allows the strong localization of the phase around two equilibrium positions, and, therefore, the abrupt character of the changes in phase. Additionally, the fluctuations establish a link between the amplitude and the phase which results in a significant enhancement of the effects of noise on the phase for small values of the amplitude. Our analysis uncovers the generality of this mechanism, and, therefore, its relevance to different contexts. Indeed, we have reported its previous characterization in studies on the appearance of selfexcited nonlinear oscillations in electronic devices [17]. Our work proves that the detected characteristics of the system response do not specifically depend on the presence of residual colored noise in the practical setup. In fact, it has been shown that the emergence of the observed features can be simply traced to broadband fluctuations. Finally, from a generalization of our approach, we have shown that the observed attenuation of the phase flips in the dynamics of a cloud of  $N$  electrons can be explained as rooted in the effective reduction of the noise strength in the center-of-mass coordinate.

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